

An integrable Hénon–Heiles system on the sphere and the hyperbolic plane

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Abstract

We construct a constant curvature analogue on the two-dimensional sphere \mathbf{S}^2 and the hyperbolic space \mathbf{H}^2 of the integrable Hénon–Heiles Hamiltonian \mathcal{H} given by

$$\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2) + \Omega(q_1^2 + 4q_2^2) + \alpha(q_1^2 q_2 + 2q_2^3),$$

where Ω and α are real constants. The curved integrable Hamiltonian \mathcal{H}_κ so obtained depends on a parameter κ which is just the curvature of the underlying space, and is such that the Euclidean Hénon–Heiles system \mathcal{H} is smoothly obtained in the zero-curvature limit $\kappa \rightarrow 0$. On the other hand, the Hamiltonian \mathcal{H}_κ that we propose can be regarded as an integrable perturbation of a known curved integrable 1 : 2 anisotropic oscillator. We stress that in order to obtain the curved Hénon–Heiles Hamiltonian \mathcal{H}_κ , the preservation of the full integrability structure of the flat Hamiltonian \mathcal{H} under the deformation generated by the curvature will be imposed. In particular, the existence of a curved analogue of the full Ramani–Dorizzi–Grammaticos (RDG) series \mathcal{V}_n of integrable polynomial potentials, in which the flat Hénon–Heiles potential can be embedded, will be essential in our construction. Such infinite family of curved RDG potentials $\mathcal{V}_{\kappa,n}$ on \mathbf{S}^2 and \mathbf{H}^2 will be also explicitly presented.

MSC: 37J35 70H06 14M17 22E60

KEYWORDS: Hénon–Heiles system, anisotropic oscillator, Ramani–Dorizzi–Grammaticos potentials, integrable systems, Lie–Poisson algebras, curvature, Poincaré disk, integrable deformation

1 Introduction

The problem considered in this paper can be stated in generic terms as follows: given a certain Liouville integrable Hamiltonian system on the two-dimensional (2D) Euclidean space

$$\mathcal{H} = \mathcal{T} + \mathcal{V} = \frac{1}{2}(p_1^2 + p_2^2) + \mathcal{V}(q_1, q_2),$$

and whose integral of the motion is given by $\mathcal{I}(p_1, p_2, q_1, q_2)$, find a one-parameter integrable generalization \mathcal{H}_κ of this system of the form

$$\mathcal{H}_\kappa = \mathcal{T}_\kappa(p_1, p_2, q_1, q_2) + \mathcal{V}_\kappa(q_1, q_2), \quad (1.1)$$

with integral of the motion given by $\mathcal{I}_\kappa(p_1, p_2, q_1, q_2)$ and fulfilling the following conditions:

- \mathcal{T}_κ has to be the kinetic energy of a particle on a 2D space with constant curvature κ , namely the sphere \mathbf{S}^2 ($\kappa > 0$), the hyperbolic space \mathbf{H}^2 ($\kappa < 0$) and the Euclidean space \mathbf{E}^2 ($\kappa = 0$). To this aim we will use the curvature-dependent formalism introduced in [1, 2, 3, 4], and the explicit form of \mathcal{T}_κ will be given afterwards (it will obviously depend upon the particular set of coordinates chosen).
- The Euclidean Hamiltonian \mathcal{H} has to be smoothly recovered in the zero-curvature limit $\kappa \rightarrow 0$, namely

$$\mathcal{H} = \lim_{\kappa \rightarrow 0} \mathcal{H}_\kappa, \quad \mathcal{I} = \lim_{\kappa \rightarrow 0} \mathcal{I}_\kappa.$$

In particular, since by construction $\mathcal{T} = \lim_{\kappa \rightarrow 0} \mathcal{T}_\kappa$, we have to impose that $\mathcal{V} = \lim_{\kappa \rightarrow 0} \mathcal{V}_\kappa$.

If these conditions are fulfilled, we shall say that \mathcal{H}_κ is a *curved \mathcal{H} system on the sphere and the hyperbolic space*. Note that, in principle, the uniqueness of this construction is not guaranteed, since different \mathcal{V}_κ potentials (and their associated \mathcal{I}_κ integrals) having the same $\kappa \rightarrow 0$ limit could be found. An example of such non-uniqueness has been explicitly given in [5, 6], where the construction of integrable curved analogues of the anisotropic oscillator potential was studied. Moreover, if the Hamiltonian \mathcal{H} is superintegrable (*i.e.* if another globally defined and functionally independent integral of the motion $\mathcal{K}(p_1, p_2, q_1, q_2)$ does exist) then we could further impose the existence of the curved (and functionally independent) analogue \mathcal{K}_κ of the second integral. If we succeed in finding such second integral, we would obtain a *superintegrable curved* generalization of \mathcal{H} .

In this respect, it is worth recalling that the search of integrable potentials with quadratic integrals of motion on \mathbf{S}^2 (and therefore admitting separation of variables) was formerly considered in [7, 8], and that a complete classification of superintegrable potentials (again with quadratic integrals) on \mathbf{S}^2 and \mathbf{H}^2 was further presented in [2, 9, 10]. More recently, trajectory isomorphisms between integrable systems on the ND sphere and the ND Euclidean space have been established as a result of central projection in [11].

We would like to stress that within the framework here presented the Gaussian curvature κ of the space enters as a deformation parameter, and the curved systems \mathcal{H}_κ can be thought of as integrable perturbations of the flat ones \mathcal{H} in terms of the curvature parameter κ . In this way, integrable Hamiltonian systems on \mathbf{S}^2 ($\kappa > 0$), \mathbf{H}^2 ($\kappa < 0$) and \mathbf{E}^2 ($\kappa = 0$) can be simultaneously constructed and analysed. This approach has been followed so far in order to construct, mainly, analogues of the oscillator and Kepler–Coulomb systems on spaces with constant curvature (see [2, 3, 4, 5, 6, 12, 13, 14, 15, 16, 17, 18, 19, 20] and references therein). Other

classical and modern results in this field, albeit without following such curvature-dependent approach, can be found in [21, 22, 23, 24, 25, 26, 27, 28, 29].

In particular, this paper is devoted to the construction of the first, to the best of our knowledge, example of a curved integrable Hénon–Heiles system in the above mentioned sense. We recall that the original Hénon–Heiles Hamiltonian [30]

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q_1^2 + q_2^2) + \lambda \left(q_1^2 q_2 - \frac{1}{3} q_2^3 \right)$$

is not Liouville-integrable, and provides an outstanding example of non-linear dynamical system that exhibits chaotic behaviour (see, for instance, [31, 32, 33]). Nevertheless, its multi-parametric generalization

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \Omega_1 q_1^2 + \Omega_2 q_2^2 + \alpha (q_1^2 q_2 + \beta q_2^3), \quad (1.2)$$

was soon proven to be Liouville integrable for three specific sets of values of the real parameters Ω_1, Ω_2 and β (see [34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45] and references therein), thus giving rise to the following distinguished families of integrable Hénon–Heiles Hamiltonians:

- The Sawada–Kotera system: $\beta = 1/3$ and $\Omega_2 = \Omega_1$, which is separable in rotated Cartesian coordinates.
- The Korteweg–de Vries (KdV) system: $\beta = 2$ with Ω_1 and Ω_2 arbitrary, which is separable in parabolic coordinates.
- The Kaup–Kuperschmidt system: $\beta = 16/3$ and $\Omega_2 = 16\Omega_1$, whose integral of the motion is quartic in the momenta.

Notice that the approach that we will follow is based on the fact that the Hénon–Heiles Hamiltonian (1.2) can be interpreted as a cubic perturbation of the anisotropic oscillator potential with frequencies $\omega_1^2 = 2\Omega_1$ and $\omega_2^2 = 2\Omega_2$. Furthermore, if we consider a particular KdV system (1.2) by setting $\Omega_2 = 4\Omega_1$ (so with $\beta = 2$), the above three integrable Hénon–Heiles Hamiltonians can be regarded, respectively, as integrable cubic perturbations of the 1 : 1, 1 : 2 and 1 : 4 superintegrable anisotropic oscillators [46, 47], namely

$$H^{\text{SK}} = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega^2(q_1^2 + q_2^2) + \alpha \left(q_1^2 q_2 + \frac{1}{3} q_2^3 \right), \quad (1.3)$$

$$H^{\text{KdV}} = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega^2(q_1^2 + 4q_2^2) + \alpha (q_1^2 q_2 + 2q_2^3), \quad (1.4)$$

$$H^{\text{KK}} = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega^2(q_1^2 + 16q_2^2) + \alpha \left(q_1^2 q_2 + \frac{16}{3} q_2^3 \right), \quad (1.5)$$

such that $\omega^2 = 2\Omega_1$.

In this paper we will present the generalization on the 2D sphere \mathbf{S}^2 and the hyperbolic (or Lobachevski) space \mathbf{H}^2 of the integrable Hénon–Heiles Hamiltonian of KdV type (1.4). We stress that the system H^{KdV} is endowed with additional and outstanding integrability properties, that will be essential in order to construct its curved analogue. In particular, the Euclidean Hamiltonian H^{KdV} (1.4) is deeply related with the so-called *Ramani–Dorizzi–Grammaticos (RDG) series* [48, 49] of integrable homogeneous polynomial potentials of degree

n on the plane that are separable in parabolic coordinates [39, 50]. They are given by

$$\mathcal{V}_n(q_1, q_2) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2i} \binom{n-i}{i} q_1^{2i} q_2^{n-2i}, \quad n = 1, 2, \dots \quad (1.6)$$

and, due to their separability, all these potentials can be freely superposed by preserving integrability. The first RDG potentials are explicitly given by

$$\begin{aligned} \mathcal{V}_1 &= 2q_2, \\ \mathcal{V}_2 &= q_1^2 + 4q_2^2, \\ \mathcal{V}_3 &= 4q_1^2 q_2 + 8q_2^3, \\ \mathcal{V}_4 &= q_1^4 + 12q_1^2 q_2^2 + 16q_2^4. \end{aligned}$$

Therefore, by considering the system

$$\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2) + \alpha_2 \mathcal{V}_2 + \alpha_3 \mathcal{V}_3$$

with $\alpha_2 = \omega^2/2$ and $\alpha_3 = \alpha/4$, we get the KdV Hénon–Heiles Hamiltonian (1.4). As a byproduct, all the elements of the RDG series (1.6) can be considered as integrable polynomial perturbations of arbitrary degree n of such Hénon–Heiles system of KdV type. Moreover, the ability of the system H^{KdV} in order to admit integrable perturbations is enhanced by the fact that certain *rational* integrable perturbations can also be superposed to the RDG series by preserving the integrability of the perturbed Hamiltonian (see [50, 51, 52, 53] and references therein). All these strong integrability properties are indeed connected with the fact that H^{KdV} is related with a certain reduction of the KdV equation [41, 51, 54, 55, 56].

It is important to stress that the strategy that we will follow in order to construct the curved Hénon–Heiles system is based on the preservation of the full integrability structure that we have already described, *i.e.*, we will consider the curved Hénon–Heiles system as an integrable perturbation of the 1 : 2 superintegrable curved oscillator already studied in [2, 5, 6], and we will assume that such curved H^{KdV} system should be also embedded within the appropriate curved RDG series of potentials.

The paper is structured as follows. The next Section is devoted to recall all the integrability features of the (Euclidean) integrable Hénon–Heiles system (1.4) and its related RDG series (1.6). In Section 3, we briefly review the two coordinate systems on \mathbf{S}^2 and \mathbf{H}^2 that will be useful in order to work out the curved version of these systems, namely, the ambient (or Weierstrass) and Beltrami (projective) canonical variables. Section 4 presents the main results of the paper. Firstly, by starting from the known curved 1 : 2 superintegrable oscillator [2, 5], the explicit expression for the full curved integrable RDG series of potentials $\mathcal{V}_{\kappa,n}$ on \mathbf{S}^2 and \mathbf{H}^2 are obtained. Secondly, the corresponding integrable curved Hénon–Heiles Hamiltonian will be defined as a suitable superposition of the $n = 2$ and $n = 3$ terms in such series. The final Section deals with some comments and open problems.

2 The flat integrable Hénon–Heiles Hamiltonian

By setting $\beta = 2$ in (1.2) we recover the KdV Hénon–Heiles Hamiltonian in terms of Cartesian canonical variables on \mathbf{E}^2 with $\{q_i, p_j\} = \delta_{ij}$, namely

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \Omega_1 q_1^2 + \Omega_2 q_2^2 + \alpha (q_1^2 q_2 + 2q_2^3). \quad (2.1)$$

We recall that in [39] this Hamiltonian was shown to be separable in terms of shifted parabolic coordinates and, furthermore, the solution of the corresponding Hamilton–Jacobi equation was also obtained. The corresponding constant of motion is quadratic in the momenta, and reads

$$I = (4\Omega_1 - \Omega_2) \left(\frac{p_1^2}{2} + \Omega_1 q_1^2 \right) + \alpha \left(p_1(q_1 p_2 - q_2 p_1) + q_1^2 \left[2\Omega_1 q_2 + \frac{\alpha}{4}(q_1^2 + 4q_2^2) \right] \right).$$

As mentioned above, the Hamiltonian (2.1) can be regarded as an *integrable cubic perturbation* of the 1 : 2 anisotropic oscillator when $\Omega_2 = 4\Omega_1 \equiv 4\Omega$, so leading to (1.4) with $\omega^2 = 2\Omega$. Indeed, such anisotropic oscillator will be our starting point in order to construct the corresponding curved generalization on \mathbf{S}^2 and \mathbf{H}^2 . Hence the explicit Hamiltonian and the integral of motion that we will consider will be

$$\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2) + \Omega(q_1^2 + 4q_2^2) + \alpha(q_1^2 q_2 + 2q_2^3), \quad (2.2)$$

$$\mathcal{I} = p_1(q_1 p_2 - q_2 p_1) + q_1^2 \left[2\Omega q_2 + \frac{\alpha}{4}(q_1^2 + 4q_2^2) \right]. \quad (2.3)$$

Now let us recall that the RDG potentials $\mathcal{V}_n(q_1, q_2)$ [48, 49] are integrable homogeneous polynomial potentials of degree n given by (1.6). It is well known that, for a given n , the integral of the motion \mathcal{L}_n for the RDG potential \mathcal{V}_n contains the \mathcal{V}_{n-1} potential. Explicitly, we have that

$$\{\mathcal{H}_n, \mathcal{L}_n\} = 0,$$

where

$$\mathcal{H}_n = \frac{1}{2}(p_1^2 + p_2^2) + \alpha_n \mathcal{V}_n, \quad \mathcal{L}_n = p_1(q_1 p_2 - q_2 p_1) + \alpha_n q_1^2 \mathcal{V}_{n-1}, \quad (2.4)$$

and the recurrence works provided that we have defined $\mathcal{V}_0 := 1$. Moreover, it is worth stressing that all the RDG potentials can be freely superposed without losing integrability [49, 52, 53], that is, the Hamiltonian

$$\mathcal{H}_{(M)} = \frac{1}{2}(p_1^2 + p_2^2) + \sum_{n=1}^M \alpha_n \mathcal{V}_n = \frac{1}{2}(p_1^2 + p_2^2) + \sum_{n=1}^M \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \alpha_n 2^{n-2i} \binom{n-i}{i} q_1^{2i} q_2^{n-2i}, \quad (2.5)$$

is endowed with a (quadratic in the momenta) integral of the motion given by

$$\begin{aligned} \mathcal{L}_{(M)} &= p_1(q_1 p_2 - q_2 p_1) + q_1^2 \sum_{n=1}^M \alpha_n \mathcal{V}_{n-1} \\ &= p_1(q_1 p_2 - q_2 p_1) + q_1^2 \left(\sum_{n=1}^M \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \alpha_n 2^{n-1-2i} \binom{n-1-i}{i} q_1^{2i} q_2^{n-1-2i} \right). \end{aligned} \quad (2.6)$$

As a consequence, the relationship among the Hamiltonian (2.2) and its integral of motion (2.3) with the RDG potentials (1.6) is immediately established through the superposition (2.5) and (2.6) by setting

$$\alpha_1 = 0, \quad \alpha_2 = \Omega, \quad \alpha_3 = \alpha/4, \quad (2.7)$$

which yields

$$\begin{aligned} \mathcal{H} &\equiv \mathcal{H}_{(3)} = \frac{1}{2}(p_1^2 + p_2^2) + \alpha_2 \mathcal{V}_2 + \alpha_3 \mathcal{V}_3, \\ \mathcal{I} &\equiv \mathcal{L}_{(3)} = p_1(q_1 p_2 - q_2 p_1) + q_1^2 (\alpha_2 \mathcal{V}_1 + \alpha_3 \mathcal{V}_2). \end{aligned}$$

Therefore, the potentials \mathcal{V}_2 and \mathcal{V}_3 define the Hénon–Heiles system, meanwhile the integral \mathcal{I} contains the linear \mathcal{V}_1 as well as the quadratic \mathcal{V}_2 RDG potentials. As we will see in the sequel, this pattern (together with an appropriate choice of the coordinates on \mathbf{S}^2 and \mathbf{H}^2) will be essential for the construction of the corresponding curved counterpart of this system.

3 Geometry and geodesic dynamics on \mathbf{S}^2 and \mathbf{H}^2

A useful description of the dynamics of integrable systems on constant curvature spaces is provided by a Lie-algebraic approach based on the following one-parameter family of 3D real Lie algebras $\mathfrak{so}_\kappa(3)$ (see [5, 6] for details):

$$[J_{12}, J_{01}] = J_{02}, \quad [J_{12}, J_{02}] = -J_{01}, \quad [J_{01}, J_{02}] = \kappa J_{12}, \quad (3.1)$$

where κ is a real parameter. The Casimir invariant of this algebra, which comes from the Killing–Cartan form, reads

$$\mathcal{C} = J_{01}^2 + J_{02}^2 + \kappa J_{12}^2. \quad (3.2)$$

Now, the 2D Riemannian spaces of constant curvature are defined as the homogeneous spaces $\mathrm{SO}_\kappa(3)/\mathrm{SO}(2)$ where $\mathrm{SO}_\kappa(3)$ is the Lie group of $\mathfrak{so}_\kappa(3)$ and $\mathrm{SO}(2)$ is the isotropy subgroup generated by J_{12} . Therefore J_{12} corresponds to the generator of rotations leaving the origin O invariant, meanwhile J_{01} and J_{02} are the generators of translations moving O along two basic directions. In this approach the parameter κ is just the Gaussian *curvature* of the space and according to its value we find:

$$\begin{array}{lll} \kappa > 0 : \text{ Sphere} & \kappa = 0 : \text{ Euclidean plane} & \kappa < 0 : \text{ Hyperbolic space} \\ \mathbf{S}^2 = \mathrm{SO}(3)/\mathrm{SO}(2) & \mathbf{E}^2 = \mathrm{ISO}(2)/\mathrm{SO}(2) & \mathbf{H}^2 = \mathrm{SO}(2,1)/\mathrm{SO}(2) \end{array}$$

The spaces \mathbf{S}^2 and \mathbf{H}^2 can be embedded in a 3D linear space $\mathbb{R}^3 = (x_0, x_1, x_2)$ where the *ambient* (or Weierstrass) coordinates must fulfil the constraint

$$\Sigma_\kappa : x_0^2 + \kappa(x_1^2 + x_2^2) = 1, \quad (3.3)$$

such that the origin is $O = (1, 0, 0) \in \mathbb{R}^3$. Now, we apply a central projection with pole $(0, 0, 0) \in \mathbb{R}^3$ from the ambient coordinates $(x_0, x_1, x_2) \in \mathbb{R}^3$ to the 2D projective space expressed in terms of *Beltrami* coordinates $(q_1, q_2) \in \mathbb{R}^2$ defined by

$$(x_0, x_1, x_2) \in \Sigma_\kappa \longrightarrow (0, 0, 0) + \mu(1, q_1, q_2) \in \Sigma_\kappa.$$

By taking into account the constraint (3.3), the projection yields the relationships between the ambient and the projective coordinates

$$x_0 = \mu = \frac{1}{\sqrt{1 + \kappa \mathbf{q}^2}}, \quad \mathbf{x} = \mu \mathbf{q} = \frac{\mathbf{q}}{\sqrt{1 + \kappa \mathbf{q}^2}}, \quad \mathbf{q} = \frac{\mathbf{x}}{x_0}, \quad (3.4)$$

where from now on we assume the notation $\mathbf{x} = (x_1, x_2)$, $\mathbf{q} = (q_1, q_2)$ and $\mathbf{p} = (p_1, p_2)$ together with

$$\mathbf{x}^2 = x_1^2 + x_2^2, \quad \mathbf{q}^2 = q_1^2 + q_2^2, \quad \mathbf{p}^2 = p_1^2 + p_2^2, \quad \mathbf{q} \cdot \mathbf{p} = q_1 p_1 + q_2 p_2.$$

In terms of the Beltrami canonical variables (\mathbf{q}, \mathbf{p}) , a symplectic realization of the Lie–Poisson analogue of the algebra $\mathfrak{so}_\kappa(3)$ (3.1) is found to be [5, 6]

$$J_{0i} = p_i + \kappa(\mathbf{q} \cdot \mathbf{p})q_i, \quad i = 1, 2; \quad J_{12} = q_1 p_2 - q_2 p_1. \quad (3.5)$$

Therefore, the free Hamiltonian \mathcal{T}_κ , providing the kinetic energy term on these spaces, is directly obtained from the Casimir (3.2) under the above realization, namely

$$\mathcal{T}_\kappa \equiv \frac{1}{2}\mathcal{C} = \frac{1}{2}(J_{01}^2 + J_{02}^2 + \kappa J_{12}^2) = \frac{1}{2}(1 + \kappa \mathbf{q}^2)(\mathbf{p}^2 + \kappa(\mathbf{q} \cdot \mathbf{p})^2), \quad (3.6)$$

and this will be the (projective) kinetic energy term that we will consider for the Ansatz (1.1). We stress that the flat/Euclidean limit $\kappa \rightarrow 0$ is always well defined in all the above expressions, thus leading to the Cartesian canonical variables on \mathbf{E}^2 used in the previous Section. In particular, when $\kappa = 0$ we have

$$x_0 = 1, \quad \mathbf{x} = \mathbf{q}, \quad J_{0i} = p_i, \quad J_{12} = q_1 p_2 - q_2 p_1, \quad \mathcal{T} = \frac{1}{2}\mathbf{p}^2.$$

We recall that a complete review of these canonical variables as well as other different ones (Poincaré and geodesic polar variables) together with the relationships among them have been presented in [6], so as to provide a complete view of several possible descriptions of the dynamics of Hamiltonian systems on \mathbf{S}^2 and \mathbf{H}^2 . Further results on geodesic motion on spaces of constant curvature can be found in [2, 4, 12, 14] (see also references therein).

4 The curved Hénon–Heiles Hamiltonian

Our construction of a curved version for the Hamiltonian (2.2) and its integral (2.3) will have as a guiding principle the preservation of the full integrability structure of the Euclidean Hénon–Heiles Hamiltonian. This implies that we have to obtain the curved counterpart $\mathcal{V}_{\kappa,n}$ of the flat RDG potentials \mathcal{V}_n (1.6). The only known initial data for this construction will be the $\mathcal{V}_{\kappa,2}$ potential, which will be given by the generalization to \mathbf{S}^2 and \mathbf{H}^2 of the Euclidean 1 : 2 anisotropic oscillator potential $\mathcal{V}_2 = q_1^2 + 4q_2^2$, which was firstly introduced in [2] and further studied in [5, 6]. As we will see in the sequel, from such curved anisotropic 1 : 2 oscillator the curved RDG series $\mathcal{V}_{\kappa,n}$ can be fully constructed in a self-consistent way, and the corresponding curved Hénon–Heiles system will be obtained as the superposition of the $n = 2$ and $n = 3$ terms from this series. Therefore, let us recall the explicit form of such curved 1 : 2 anisotropic oscillator.

Proposition 1. [2, 5] *Let $\mathcal{H}_\kappa^{1:2}$ be the Hamiltonian*

$$\mathcal{H}_\kappa^{1:2} = \mathcal{T}_\kappa + \mathcal{V}_\kappa^{1:2} = \frac{1}{2}(1 + \kappa \mathbf{q}^2)(\mathbf{p}^2 + \kappa(\mathbf{q} \cdot \mathbf{p})^2) + \Omega \frac{q_1^2(1 + \kappa q_2^2) + 4q_2^2}{(1 - \kappa q_2^2)^2}, \quad (4.1)$$

which is written in terms of Beltrami coordinates (3.4) and their conjugate momenta. This system is superintegrable for any value of κ and Ω , and the two functionally independent integrals of motion for $\mathcal{H}_\kappa^{1:2}$ are

$$\mathcal{I}_\kappa^{1:2} = \frac{1}{2}(J_{01}^2 + \kappa J_{12}^2) + \Omega \frac{q_1^2(1 + \kappa q_2^2)}{(1 - \kappa q_2^2)^2}, \quad \mathcal{L}_\kappa^{1:2} = J_{01} J_{12} + 2\Omega \frac{q_1^2 q_2}{(1 - \kappa q_2^2)^2},$$

where J_{01} , J_{12} are the functions given by (3.5).

In the Euclidean space with $\kappa = 0$ the above expressions are reduced to the well-known flat ones, namely

$$\begin{aligned} \mathcal{H}^{1:2} &= \mathcal{T} + \mathcal{V}^{1:2} = \frac{1}{2}\mathbf{p}^2 + \Omega(q_1^2 + 4q_2^2), \\ \mathcal{I}^{1:2} &= \frac{1}{2}p_1^2 + \Omega q_1^2, \quad \mathcal{L}^{1:2} = p_1(q_1 p_2 - q_2 p_1) + 2\Omega q_1^2 q_2. \end{aligned} \quad (4.2)$$

Hence $\mathcal{I}_\kappa^{1:2}$ is the curved generalization of the integral $\mathcal{I}^{1:2}$, which comes from the separability of the flat system in Cartesian coordinates, meanwhile the additional integral $\mathcal{L}_\kappa^{1:2}$ ensures superintegrability. Note that the expressions (4.2) can be directly written in terms of the RDG potential (1.6) and its integrals of motion (2.4) with $n = 2$ as

$$\begin{aligned}\mathcal{H}^{1:2} &\equiv \mathcal{H}_2 = \frac{1}{2} \mathbf{p}^2 + \alpha_2 \mathcal{V}_2, & \mathcal{V}^{1:2} &\equiv \mathcal{V}_2 = q_1^2 + 4q_2^2, \\ \mathcal{L}^{1:2} &\equiv \mathcal{L}_2 = p_1(q_1 p_2 - q_2 p_1) + \alpha_2 q_1^2 \mathcal{V}_1,\end{aligned}$$

where $\mathcal{V}_1 = 2q_2$ and $\alpha_2 = \Omega$.

Our Ansatz will be that this integrability structure in terms of \mathcal{V}_1 and \mathcal{V}_2 should be preserved in the curved case for a suitably defined curved RDG Hamiltonian, $\mathcal{H}_{\kappa,n}$. Therefore such Hamiltonian must be integrable for any value of n . With this in mind we can rewrite the expressions from Proposition 1 in the form

$$\begin{aligned}\mathcal{H}_\kappa^{1:2} &\equiv \mathcal{H}_{\kappa,2} = \mathcal{T}_\kappa + \alpha_2 \mathcal{V}_{\kappa,2}, & \mathcal{V}_{\kappa,2} &= \frac{q_1^2(1 + \kappa q_2^2) + 4q_2^2}{(1 - \kappa q_2^2)^2}, \\ \mathcal{L}_\kappa^{1:2} &\equiv \mathcal{L}_{\kappa,2} = J_{01} J_{12} + \alpha_2 \frac{q_1^2}{1 + \kappa \mathbf{q}^2} \mathcal{V}_{\kappa,1},\end{aligned}$$

so that $\alpha_2 \mathcal{V}_{\kappa,2} = \mathcal{V}_\kappa^{1:2}$ (4.1) and where the curved RDG potential with $n = 1$ would be

$$\mathcal{V}_{\kappa,1} = \frac{2q_2(1 + \kappa \mathbf{q}^2)}{(1 - \kappa q_2^2)^2}.$$

Now, by taking into account (2.4) we can also propose the following curved expression for the RDG Hamiltonian when $n = 1$:

$$\mathcal{H}_{\kappa,1} = \mathcal{T}_\kappa + \alpha_1 \mathcal{V}_{\kappa,1},$$

whose integral of the motion is easily shown to be

$$\mathcal{L}_{\kappa,1} = J_{01} J_{12} + \alpha_1 \frac{q_1^2}{1 + \kappa \mathbf{q}^2} \mathcal{V}_{\kappa,0}.$$

where the curved version of the constant (0-th order) RDG potential $\mathcal{V}_0 = 1$ would be

$$\mathcal{V}_{\kappa,0} := \frac{(1 + \kappa q_2^2)(1 + \kappa \mathbf{q}^2)}{(1 - \kappa q_2^2)^2}. \quad (4.3)$$

Surprisingly enough, this means that the self-consistency conditions that guarantee the existence of a curved RDG series lead to the fact that $\mathcal{V}_{\kappa,0}$ is no longer a constant function. In fact, from these results the form of the full curved RDG series for arbitrary n can be obtained by induction, and is the following:

Proposition 2. *The curved RDG potentials on the sphere \mathbf{S}^2 and the hyperbolic space \mathbf{H}^2 are defined in Beltrami coordinates by*

$$\mathcal{V}_{\kappa,n} = \left(\frac{1 + \kappa \mathbf{q}^2}{1 - \kappa q_2^2} \right)^2 \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2i} \binom{n-i}{i} \left(\frac{q_1}{\sqrt{1 + \kappa \mathbf{q}^2}} \right)^{2i} \left(1 - \frac{i}{n-i} \left[\frac{\kappa q_1^2}{1 + \kappa \mathbf{q}^2} \right] \right) \left(\frac{q_2}{1 + \kappa \mathbf{q}^2} \right)^{n-2i} \quad (4.4)$$

Table 1: The potentials in the RDG series for $n = 0, 1, \dots, 4$ on \mathbf{E}^2 in Cartesian coordinates \mathbf{q} (1.6) and their curved counterpart on \mathbf{S}^2 and \mathbf{H}^2 in ambient coordinates (4.5) such that $x_0^2 + \kappa \mathbf{x}^2 = 1$.

\mathbf{E}^2 : Cartesian coordinates \mathbf{q}	\mathbf{S}^2 and \mathbf{H}^2 : Ambient coordinates (x_0, \mathbf{x})
$\mathcal{V}_0 = 1$	$\mathcal{V}_{\kappa,0} = \frac{1 - \kappa x_1^2}{(x_0^2 - \kappa x_2^2)^2}$
$\mathcal{V}_1 = 2q_2$	$\mathcal{V}_{\kappa,1} = \frac{2x_0x_2}{(x_0^2 - \kappa x_2^2)^2}$
$\mathcal{V}_2 = q_1^2 + 4q_2^2$	$\mathcal{V}_{\kappa,2} = \frac{x_1^2(1 - \kappa x_1^2) + 4x_0^2x_2^2}{(x_0^2 - \kappa x_2^2)^2}$
$\mathcal{V}_3 = 4q_1^2q_2 + 8q_2^3$	$\mathcal{V}_{\kappa,3} = \frac{4x_0x_1^2x_2(1 - \frac{1}{2}\kappa x_1^2) + 8x_0^3x_2^3}{(x_0^2 - \kappa x_2^2)^2}$
$\mathcal{V}_4 = q_1^4 + 12q_1^2q_2^2 + 16q_2^4$	$\mathcal{V}_{\kappa,4} = \frac{x_1^4(1 - \kappa x_1^2) + 12x_0^2x_1^2x_2^2(1 - \frac{1}{3}\kappa x_1^2) + 16x_0^4x_2^4}{(x_0^2 - \kappa x_2^2)^2}$
\vdots	\vdots

with $n \in \mathbb{N}^+$. The curved RDG Hamiltonian

$$\mathcal{H}_{\kappa,n} = \mathcal{T}_\kappa + \alpha_n \mathcal{V}_{\kappa,n},$$

is integrable with a constant of motion quadratic in the momenta, since the function

$$\mathcal{L}_{\kappa,n} = J_{01}J_{12} + \alpha_n \frac{q_1^2}{1 + \kappa \mathbf{q}^2} \mathcal{V}_{\kappa,n-1},$$

fulfils $\{\mathcal{H}_{\kappa,n}, \mathcal{L}_{\kappa,n}\} = 0$. Here $\mathcal{V}_{\kappa,0}$ is given by (4.3) and J_{01} , J_{12} and \mathcal{T}_κ are the functions given by (3.5) and (3.6).

Notice that under the flat limit $\kappa \rightarrow 0$ the above expressions straightforwardly lead to the Euclidean \mathcal{V}_n potential (1.6), and to \mathcal{H}_n and \mathcal{L}_n (2.4) together with $\mathcal{V}_0 = 1$. On the other hand, the curved RDG potentials (4.4) can also be straightforwardly written in terms of the ambient coordinates (x_0, \mathbf{x}) by applying (3.4). They read

$$\mathcal{V}_{\kappa,n} = \frac{1}{(x_0^2 - \kappa x_2^2)^2} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2i} \binom{n-i}{i} x_1^{2i} \left(1 - \frac{i}{n-i} \kappa x_1^2\right) (x_0 x_2)^{n-2i}, \quad n \in \mathbb{N}^+. \quad (4.5)$$

This expression facilitates the translation of such potentials in any other coordinate system. In Table 1 we illustrate all these results for $n = 0, 1, \dots, 4$. Recall that when $\kappa = 0$ the ambient coordinates are just $x_0 = 1$ and $(x_1, x_2) = (q_1, q_2)$.

The next step consists in demonstrating that, as in the Euclidean case, the curved RDG potentials can be freely superposed, so generalizing to the curved case the expressions (2.5) and (2.6). This result can be obtained through straightforward computations and reads:

Theorem 3. *For any $M \in \mathbb{N}^+$, the Hamiltonian formed by the superposition of the curved RDG potentials (4.4) and given by*

$$\mathcal{H}_{\kappa,(M)} = \mathcal{T}_\kappa + \sum_{n=1}^M \alpha_n \mathcal{V}_{\kappa,n},$$

Poisson-commutes with the function

$$\mathcal{L}_{\kappa,(M)} = J_{01}J_{12} + \frac{q_1^2}{1 + \kappa \mathbf{q}^2} \sum_{n=1}^M \alpha_n \mathcal{V}_{\kappa,n-1}.$$

Consequently, we have obtained all the ingredients needed in order to obtain the curved counterpart of the Hénon–Heiles Hamiltonian (2.2) and its integral (2.3), since Proposition 2 provides the new curved ‘cubic’ potential $\mathcal{V}_{\kappa,3}$, while Theorem 3 establishes the appropriate integrable superposition with the known curved ‘quadratic’ term $\mathcal{V}_{\kappa,2}$. In this way we find, through the identification of parameters given by (2.7), the following main result of the paper, which gives the expression for an integrable curved Hénon–Heiles Hamiltonian of KdV type that preserves the full integrability structure of the flat Hamiltonian system.

Proposition 4. *The curved counterpart of the Hénon–Heiles Hamiltonian (2.2) on \mathbf{S}^2 and \mathbf{H}^2 is written in Beltrami variables as*

$$\begin{aligned} \mathcal{H}_\kappa &= \mathcal{T}_\kappa + \mathcal{V}_\kappa = \mathcal{T}_\kappa + \alpha_2 \mathcal{V}_{\kappa,2} + \alpha_3 \mathcal{V}_{\kappa,3} \\ &= \frac{1}{2} (1 + \kappa \mathbf{q}^2) (\mathbf{p}^2 + \kappa (\mathbf{q} \cdot \mathbf{p})^2) + \Omega \frac{q_1^2 (1 + \kappa q_2^2) + 4q_2^2}{(1 - \kappa q_2^2)^2} \\ &\quad + \alpha \frac{q_1^2 q_2 (1 + \kappa \mathbf{q}^2 - \frac{1}{2} \kappa q_1^2) + 2q_2^3}{(1 - \kappa q_2^2)^2 (1 + \kappa \mathbf{q}^2)} \end{aligned}$$

and Poisson-commutes with the function

$$\begin{aligned} \mathcal{I}_\kappa &= J_{01}J_{12} + \frac{q_1^2}{1 + \kappa \mathbf{q}^2} (\alpha_2 \mathcal{V}_{\kappa,1} + \alpha_3 \mathcal{V}_{\kappa,2}) \\ &= (p_1 + \kappa (\mathbf{q} \cdot \mathbf{p}) q_1) (q_1 p_2 - q_2 p_1) \\ &\quad + \frac{q_1^2}{1 + \kappa \mathbf{q}^2} \left(\Omega \frac{2q_2 (1 + \kappa \mathbf{q}^2)}{(1 - \kappa q_2^2)^2} + \alpha \frac{q_1^2 (1 + \kappa q_2^2) + 4q_2^2}{4(1 - \kappa q_2^2)^2} \right), \end{aligned}$$

such that $\alpha_2 = \Omega$ and $\alpha_3 = \alpha/4$.

Of course, the vanishing curvature $\kappa \rightarrow 0$ limit of these two expressions is, respectively, (2.2) and (2.3). Moreover, Theorem 3 can now be understood as the tool that provides further integrable deformations of this curved Hénon–Heiles system whose flat limit is a polynomial potential of degree M . On the other hand, the expression of the curved Hénon–Heiles potential \mathcal{V}_κ in terms of ambient coordinates can be easily derived from Table 1.

5 Concluding remarks and open problems

In this paper we have presented the generalization on 2D spaces with constant curvature of the 1 : 2 integrable Hénon–Heiles system of KdV type. This new Liouville integrable Hamiltonian system has been constructed by following a ‘curved integrability criterium’, *i.e.*, by exploring all the integrability properties of the Euclidean system \mathcal{H} and by constructing its curved analogue \mathcal{H}_κ by preserving the full integrability structure of the flat case. In this way, by starting from the known 1 : 2 curved superintegrable anharmonic oscillator, the full curved RDG series of integrable potentials have been constructed, and the Hénon–Heiles system can

thus be obtained as the appropriate superposition of the second- and third-order curved RDG potentials. We stress that the fact that the latter series of potentials can be expressed as rational functions of the (projective) Beltrami coordinates turns out to be very helpful from the computational viewpoint (see also [28] and references therein for a recent review on the various facets of the projective dynamics approach).

This result suggests in a natural way several open problems that we plan to face in the near future through the same approach. The first one would be the construction of the curved analogue of the Hénon–Heiles system of KdV type with arbitrary anisotropic oscillator frequencies ($\beta = 2$ and (Ω_1, Ω_2) arbitrary in (1.2)), as well as the study of the generalization to the curved case of the integrable rational perturbations of the KdV flat Hénon–Heiles system introduced in [50, 51, 52, 53].

The following natural step would consist in the construction of the curved Sawada–Kotera Hénon–Heiles system (1.3) ($\beta = 1/3$ and $\Omega_1 = \Omega_2$) by starting from the also known superintegrable curved 1 : 1 isotropic (Higgs) oscillator (see [5, 23]). Since the flat Sawada–Kotera system is separable in rotated Cartesian coordinates, an associated series of integrable polynomial perturbations does also exist, and the curved Sawada–Kotera system should be obtained by following a similar procedure to the one presented in this paper. In the same manner, the curved Kaup–Kuperschmidt Hénon–Heiles system (1.5) ($\beta = 16/3$ and $\Omega_2 = 16 \Omega_1$) should be obtained by starting from a superintegrable curved 1 : 4 oscillator whose integral of the motion should be quartic in the momenta, but in this case even the latter curved anharmonic oscillator is still unknown. Finally, the construction of the quantum integrable version of all these new curved systems as well as the study of their quantum dynamics constitute a challenging problem. In this respect, we recall that the preservation of the integrability properties under the deformation induced by the curvature turns out to be essential in order to guarantee the exact solvability of the corresponding quantum system [57].

Acknowledgements

This work was partially supported by the Spanish Ministerio de Economía y Competitividad (MINECO) under grants MTM2010-18556 and MTM2013-43820-P, and by the Spanish Junta de Castilla y León under grant BU278U14. The authors acknowledge the referees for their valuable reports.

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